Functional Calculi of Families of Matrices and Operators

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There are many ways to form a function f of a matrix L, or more generally of an operator L in a Banach space $\mathcal{X}$. If f is the polynomial $f(z) = \sum_{k=0}^{N} c_k z^k$, then just define $f(L) = \sum_{k=0}^{N} c_k L^k$. Slightly more generally, suppose a holomorphic function f is defined by $f(z) = \sum_{k=0}^{\infty} c_k z^k$ when $|z| < R$, where the radius of convergence R is larger than the norm $||L||$ of L. Then set $f(L) = \sum_{k=0}^{\infty} c_k L^k$. Much more general is the Riesz–Dunford functional calculus. Suppose f is a holomorphic function defined on a neighbourhood Ω of the spectrum $\sigma(L)$ of L. The spectrum is the compact set of all λ in the complex plane $\mathbb{C}$ for which the resolvent $R_L(\lambda) = (\lambda I - L)^{-1}$ does not exist as a bounded operator on $\mathcal{X}$. (For a matrix L, this is just the set of eigenvalues of L.) Then define $f(L) = \frac{1}{2\pi i}\int_\gamma R_L(\zeta) f(\zeta)\, d\zeta$ where γ is a curve in Ω which winds anti-clockwise around $\sigma(L)$. These definitions are all consistent, they have natural properties such as $(fg)(L) = f(L)g(L)$ and $\sigma(f(L)) = f(\sigma(L))$, and moreover are useful in a variety of contexts. A particular mapping $f \mapsto f(L)$ is called a functional calculus of L.

When $\mathbf{L} = (L_1, L_2, \dots L_n)$ is a commuting family of matrices or operators, then it is clear how to define $f(\mathbf{L})$ when f is a polynomial or power series in n variables. However there are various approaches to defining a joint spectrum $\sigma(\mathbf{L})$ of L, and to constructing a functional calculus of $\mathbf{L}$. When the operators satisfy the property that $\sigma(\sum \xi_j L_j) \subset \mathbb{R}$ for all $\xi \in \mathbb{R}^n$, then one possibility is to use Clifford algebras, and to replace the use of the Cauchy integral in the Riesz–Dunford functional calculus by a higher dimensional Clifford–Cauchy integral.

When the operators L_j do not commute with one another, then it becomes more difficult to construct a reasonable theory. Nevertheless, this is important. For example, the Weyl functional calculus was introduced to consider functions of position (Q_j) and momentum (P_j) under the canonical commutation relations $Q_j P_j - P_j Q_j = i\hbar I$ of quantum theory. In the Weyl functional calculus, one takes symmetric products in forming polynomials. For example, if $\mathbf{L} = (L_1, L_2)$ and $f(x_1, x_2) = x_1 x_2$, then $f(\mathbf{L}) = \frac{1}{2}(L_1 L_2 + L_2 L_1)$. The Clifford approach generalizes quite well to this context, though not easily, and gives us the opportunity to extend the Weyl functional calculus to more general situations, and to identify a type of spectral set as the support of the functional calculus.

A seminal paper on commuting operators is: Alan McIntosh and Alan Pryde, *A functional calculus for several commuting operators*, Indiana University Math. Journal **36** (1987), 421–439; while a seminal paper on non-commuting operators is: Brian Jefferies, Alan McIntosh and James Picton–Warlow, *The monogenic functional calculus*, Studia Mathematica **136** (1999), 99–119.

A readable book on this material which contains the background and all the latest results is: Brian Jefferies, *Spectral properties of noncommuting operators*, Lecture Notes in Mathematics **1843**, Springer–Verlag, 2004.