MAT-51327 Numerical methods for partial differential equations Exercise 1
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1. Define a piecewise linear function $f : \mathbb{R} \to \mathbb{R}$ as follows:

$$f(t) = \begin{cases} t & \text{for } 0 \leq t < \frac{1}{2}, \\ 1-t & \text{for } \frac{1}{2} \leq t < 1, \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Compute the weak derivative $f'(t)$.

   (b) Show that $f \in H^1(\mathbb{R})$.

2. Let $\mathcal{H}$ be a Hilbert space with scalar field $\mathbb{R}$, and denote the inner product by $\langle \cdot , \cdot \rangle$. Fix $x \in \mathcal{H}$. Show that $L_x : \mathcal{H} \to \mathbb{R}$ defined by $L_x(y) = \langle x, y \rangle$ is a bounded linear functional.

3. Let $\mathcal{H}$ be a Hilbert space with scalar field $\mathbb{R}$. Assume that $L : \mathcal{H} \to \mathbb{R}$ is a bounded linear functional. Denote

$$M = \{y \in \mathcal{H} \,:\, L(y) = 0\}.$$

   The *orthogonal complement* of the set M is defined by

$$M^\perp := \{z \in \mathcal{H} \,:\, \langle z, x \rangle = 0 \text{ for every } x \in M\}.$$

   Show that $M^\perp$ is a vector space of dimension 1 (unless $M = \mathcal{H}$).

4. Prove the Riesz representation theorem in a Hilbert space with scalar field $\mathbb{R}$. (You can follow the course textbook but you need to explain the steps in the proof in order to earn an exercise point.)