MAT-52500 Inverse Problems, fall 2007 Interim exam (10.10.2007)
Samuli Siltanen

1. A problem of the form *find* x*, when* $m = Ax + \epsilon$ *is given* is called an inverse problem if it is *not* well-posed in the sense of Hadamard.

   (a) Explain Hadamard's definition of a well-posed problem.

   (b) What are the crucial properties of the matrices U, D and V in the singular value decomposition $A = UDV^T$?

   (c) Give the formula for truncated SVD solution.

2. Consider the Tikhonov solution $x_\delta = \arg\min_x \{\|Ax - m\|^2 + \delta\|L(x - x^*)\|^2\}$.

   (a) What is the role of the term $\|Ax - m\|^2$?

   (b) What are the roles of L and x^*?

   (c) Write the problem in stacked form.

3. In total variation regularization we take a difference matrix L and consider the minimization problem

$$\arg\min_{x \in \mathbb{R}^n} \Big(\|Ax - m\|_2^2 + \delta\|Lx\|_1 \Big). \tag{1}$$

   (a) What is the goal of total variation regularization?

   (b) Show that (1) is equivalent to the problem

$$\arg\min_x \Big(\|Ax\|_2^2 - 2m^T Ax + \beta \underline{\mathbf{1}}^T V_+ + \beta \underline{\mathbf{1}}^T V_- \Big) \tag{2}$$

   with suitable definition of $\beta, \mathbf{1}, V_+$ and V_-. Hint: write $Lx = V_+ - V_-$.

   (c) Convert (2) to quadratic standard form $\arg\min_y \{\frac{1}{2} y^T H y + f^T y\}$ with appropriate definition of y and equality and inequality restraints.

4. Consider the measurement model $m = Ax + \epsilon$, where the size of A is $n \times n$.

   (a) Assume that each component of the noise vector ε is Cauchy distributed: $\varepsilon_j \sim \pi^{-1}(1 + \varepsilon_j^2)^{-1}$. Write down the likelihood distribution $p(m|x)$.

   (b) Assume that we know *a priori* that the components of the vector x are non-negative and less than 1. Define a prior distribution expressing this information.

   (c) Write the posterior distribution $p(x|m)$ using Bayes formula and the prior and likelihood distributions in (a) and (b). (Normalization constant is not needed.)