1. A problem of the form *find x, when $m = Ax + \epsilon$ is given* is called an inverse problem if it is *not* well-posed in the sense of Hadamard.

   (a) Formulate Hadamard's definition of a well-posed problem.

   (b) Give an example of an inverse problem arising from engineering or physics. Explain why Hadamard's conditions are violated in the example.

2. Consider the Tikhonov solution $x_\delta = \arg\min_x\{\|Ax - m\|^2 + \delta\|Lx\|_2^2\}$.

   (a) What is the role of the term $\|Ax - m\|^2$?

   (b) Give two examples of typical choices of matrix L.

   (c) Write the problem in stacked form.

3. In TV regularization one takes a finite difference matrix L and solves

$$\arg\min_{x\in\mathbb{R}^n}\Big(\|Ax - m\|_2^2 + \delta\|Lx\|_1\Big). \qquad (1)$$

   (a) Write problem (1) in an approximate form by redefining the absolute value function so that the objective function becomes differentiable.

   (b) Describe some algorithm for solving the minimization problem (a). (Suitable choices include steepest descent and Barzilai-Borwein methods.)

4. In statistical inversion the posterior distribution $p(x|m) \sim p(x)p(m|x)$ is the complete answer to the inverse problem.

   (a) Give the definition of the maximum a posteriori (MAP) estimate for x.

   (b) Assume $m = Ax + \epsilon$ with Gaussian noise ϵ. Take $p(x) \sim \exp(-\delta\|Lx\|_2^2)$. Show that finding the MAP estimate is equivalent to generalized Tikhonov regularization.

5. Consider again the posterior distribution $p(x|m) \sim p(x)p(m|x)$.

   (a) Give the definition of the conditional mean estimate for x.

   (b) Describe some numerical method that can be used to (approximately) compute the conditional mean estimate.