Computational aspects of Inverse Problems — Exercise 1
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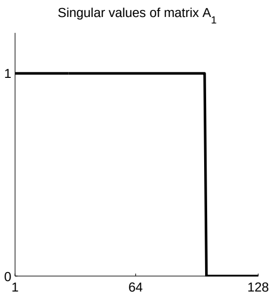
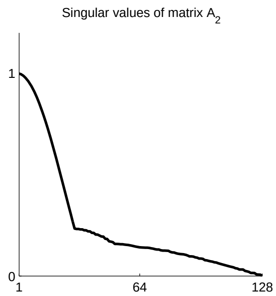
1. Let $x \in \mathbb{R}^8$ be a signal and $p = [p_1\, p_2\, p_3]^T$ a point spread function. Write down the 8×8 matrix A modeling the one-dimensional convolution $p * x$. Use the periodic boundary condition $x_j = x_{j+8}$ for all $j \in \mathbb{Z}$.

2. If matrices A_1 and A_2 have the singular values shown below, what conditions of Hadamard do they violate, if any?





3. Denote the columns of a $n \times n$ matrix V by $V_1, \ldots, V_n$. Show that $Vz = \sum_{j=1}^{n} z_j V_j$.

4. Consider equations $x_1 + x_2 = 1$, $x_2 = -2$ and $-\frac{1}{3}x_1 + x_2 = -2$.

   (a) Write the equations in the matrix form $Ax = y$. (That is, specify the elements in the 3×2 matrix A and the vector $y \in \mathbb{R}^3$.)

   (b) Use Matlab to compute the singular value decomposition $A = UDV^T$.

   (c) Using the result of (b), construct D^+ and the minimum norm solution $x^+ := VD^+U^Ty$ in Matlab. Draw the three lines specified by the equations and the point x^+ in the (x_1, x_2)-plane. Discuss the result.

5. Download the Matlab routines `ex_conv1Ddata_comp.m` and `ex_conv1D_naive.m` from the course web page to your working directory. Create a new folder called `data`. Modify the two routines to study the following problems.

   Consider a continuous point spread function $p : \mathbb{R} \to \mathbb{R}$ given by

$$p(s) = \begin{cases} 1 - s & \text{for } 0 \leq s \leq 1, \\ 1 + s & \text{for } -1 \leq s < 0, \\ 0 & \text{otherwise.} \end{cases}$$

   Choose any function $x : \mathbb{R} \to \mathbb{R}$ satisfying $x(s) = 0$ for $s < 0$ and $s > 10$.

   (a) Create data with `ex_conv1Ddata_comp.m` and try to reconstruct your signal with `ex_conv1D_naive.m` using the same discretization in both files. Do you see unrealistically good reconstructions? What happens when the discretization is refined?

   (b) Repeat (a) using finer discretization for generating data. Can you avoid inverse crime?

6. Continue the previous problem. Write a Matlab routine that plots reconstructions using truncated SVD using $1, 2, 3, \ldots$ singular vectors. Observe the evolution of reconstruction when the number of singular vectors grows.