Computational Aspects of Inverse Problems — Exercise 2

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1. Show that $A^TA + \delta I$ is an invertible matrix if $\delta > 0$. Hint: use SVD.

2. Show that the variational form corresponding to the minimization problem

$$T_\delta(m) = \arg\min_{z\in\mathbb{R}^n}\{\|Az - m\|^2 + \delta\|Lz\|^2\} \tag{1}$$

is given by $\big\langle (A^TA + \delta L^TL)T_\delta(m) - A^Tm, w\big\rangle = 0$ for all $w \in \mathbb{R}^n$.

3. Write the generalized Tikhonov problem

$$T_\delta(m) = \arg\min_{z\in\mathbb{R}^n}\{\|Az - m\|^2 + \delta\|L(z - x_\star)\|^2\}$$

in stacked form.

4. Continue problems 5 and 6 of previous exercises. Find the optimal values for δ using the L-curve method and noise levels 1% and 10%. (You can do this by modifying `ex_conv1D_Tikhonov_Lcurve` available at the course web page.)

5. Continue problem 6 of last week's exercises.

   (a) Construct a matrix L that discretizes the second derivative d^2/ds^2.

   (b) Using 1% noise level and formula (1), find the δ value that gives the smallest relative error. (You can do this by modifying `ex_conv1D_stackedform.m` available at the course web page.)

   (c) Simulate data from a smooth example signal instead of the discontinuous used so far. Repeat (b). Can you get a lower minimal error now that the smoothness assumption built in (1) is valid?

6. Two-dimensional convolution of $K \times L$ pixel images X is explained in Section 2.2.2 of the lecture notes. Write total variation regularization

$$\arg\min_{z\in\mathbb{R}^n}\left\{\|Ax - m\|_2^2 + \delta\sum_{k=1}^{K}\sum_{\ell=1}^{L-1}|X_{k(\ell+1)} - X_{k\ell}| + \delta\sum_{k=1}^{K-1}\sum_{\ell=1}^{L}|X_{(k+1)\ell} - X_{k\ell}|\right\}$$

in the standard quadratic form

$$\arg\min_y\left\{\frac{1}{2}y^THy + f^Ty\right\}$$

with appropriate equality and inequality constraints. Note that the relation between x and X is (in MATLAB notation) `x = X(:)`.