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Note that the exercises continue on page 2!

1. Two-dimensional convolution of $K \times L$ pixel images X is explained in Section 2.2.2 of the lecture notes. Write total variation regularization

$$\arg\min_{z\in\mathbb{R}^n}\left\{\|Ax-m\|_2^2+\delta\sum_{k=1}^{K}\sum_{\ell=1}^{L-1}|X_{k(\ell+1)}-X_{k\ell}|+\delta\sum_{k=1}^{K-1}\sum_{\ell=1}^{L}|X_{(k+1)\ell}-X_{k\ell}|\right\}$$

in the standard quadratic form

$$\arg\min_{y}\left\{\frac{1}{2}y^THy+f^Ty\right\}$$

with appropriate equality and inequality constraints. Note that the relation between x and X is (in MATLAB notation) `x` = `X(:)`.

2. Modify the files `bb_deblur.m`, `db_aTV.m`, `db_aTV_feval.m`, `db_aTV_fgrad.m`, `db_aTV_grad.m`, `db_misfit.m` and `db_misfit_grad.m` and solve the 1D deconvolution problem using Barzilai-Borwein minimization and approximate total variation regularization. Choose the first steplength simply to be small enough.

3. Compute the value of π approximately as follows. Sample N points $(x_j, y_j) \in [0,1]^2$, $j = 1,\ldots,N$, by drawing each x_j and y_j independently from uniform distribution on $[0,1]$ using Matlab's `rand` command. Then we have

$$\pi \approx 4|\{j\,:\,x_j^2+y_j^2<1\}|/N,$$

where $|B|$ denotes the number of elements in the set B. Why? Find out (by experimentation) roughly how big must N be for us to get the digits 3.14 correct more than half of the time we run the algorithm?

4. Consider the measurement $m = Ax + \varepsilon$, where the noise components ε_j are independent and have Gaussian probability density function

$$\frac{1}{\sigma\sqrt{2\pi}} \exp(-\frac{1}{2\sigma^2}\varepsilon_j^2).$$

   (a) Write down the likelihood distribution $p(m|x)$.

   (b) Take $p(x) = C\exp(-\alpha\|Lx\|_2^2)$ as prior distribution, where L is the difference matrix used in last week's exercise 5. Write down the posterior distribution (without normalizing).

   (c) Show that finding the MAP estimate for the posterior distribution in (b) is equivalent to Tikhonov regularization with suitable relationship between σ, δ and α.

5. Continue last week's exercise 5, this time with dimension $n = 32$. Compute the *same* reconstruction in two ways: first by computing the regularized solution with last week's code from 5(b), and second by using conditional mean estimate and Metropolis-Hastings method. (This goal is based on the facts that the MAP estimate coincides with the conditional mean estimate in this Gaussian case and that according to exercise 4 above the MAP estimate coincides with the Tikhonov regularized solution.) Proceed as follows.

   (a) Choose the candidate in Metropolis-Hastings algorithm by

$$x_c = x^{(\ell)} + \rho \cdot \texttt{randn}(\texttt{size}(x^{(\ell)}))$$

   using standard deviation $\rho > 0$. Try to find such value for ρ that 20–40% of the candidates are accepted. Can you reach 10% relative error with this computation? If so, how many samples are needed for that?

   (b) Try updating only some components of $x^{(\ell)}$ at a time. That is, when constructing a candidate, first choose randomly 8 dimensions $j_1, j_2, \ldots, j_8$ out of the available 32. Then define the candidate by setting

$$(x_c)_{j_\nu} = (x^{(\ell)})_{j_\nu} + \rho \cdot \texttt{randn}$$

   for each $\nu = 1, \ldots, 8$. Again, try to find such value for ρ that 20–40% of the candidates are accepted. Can you reach 10% relative error with this computation? If so, how many samples are needed for that?

6. Compute the CM estimate as in exercise 5 above but including nonnegativity constraint. Do this simply by rejecting any candidate that has at least one negative element.